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Coherent states for angular momentum

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Abstract. Angular momentum states analogous to the coherent states of the harmonic oscillator are defined and their properties discussed.

1. Introduction

The coherent state was first constructed (Schrödinger 1926, Glauber 1963) for the simple harmonic oscillator. The Hamiltonian of the system

$$H = p^2/2m + m\omega^2 x^2/2, \tag{1}$$

may be rewritten as

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}), \tag{2}$$

by defining annihilation and creation operators

$$a = (p - \mathrm{i}m\omega x)/(2m\omega\hbar)^{1/2}, \qquad a^\dagger = (p + \mathrm{i}m\omega x)/(2m\omega\hbar)^{1/2}. \tag{3}$$

The eigenstates of the Hamiltonian, $|n\rangle$, belonging to the energy eigenvalue

$$E_n = \hbar\omega(n + \tfrac{1}{2}),$$

where n is a non-negative integer, may then be obtained with the properties

$$a^\dagger a|n\rangle = n|n\rangle, \qquad a^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle, \qquad a|n\rangle = n^{1/2}|n-1\rangle. \tag{4}$$

The coherent state may then be constructed out of these states, namely

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{5}$$

where α is a complex number, and the factor outside the summation sign is the normalization constant. The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator, namely,

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{6}$$

The coherent state may also be written in the form

$$|\alpha\rangle = \exp(-\alpha^* a + \alpha a^\dagger)|0\rangle, \tag{7}$$

and is thus a 'displacement of the vacuum'. The coherent states form a complete (albeit an over-complete) set in the sense that

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \mathbb{1}, \tag{8}$$

where the integration is over the entire complex α plane. The coherent state constitutes a state of minimum uncertainty, namely,

$$\Delta p \Delta x = \hbar/2. \tag{9}$$

Also the coherent state, a non-stationary state, develops with time (taking $\alpha(t = 0) = \lambda\, e^{-i\theta}$) yielding

$$\langle\alpha, t|x|\alpha, t\rangle = \left[-2\lambda\left(\frac{\hbar}{2m\omega}\right)^{1/2}\right]\sin(\omega t+\theta). \tag{10}$$

Identifying the constant in square brackets with the amplitude (in the limit $\hbar \to 0$, $\lambda \to \infty : \lambda\sqrt{\hbar} \to$ finite limit), the expectation value of the displacement in the coherent state behaves like the displacement of a classical oscillator. In this sense the coherent state is called a 'classical state'.

The object of the present work is to show that coherent states may be constructed for an angular momentum system and the points of similarity and dissimilarity with the properties of the oscillator coherent states, discussed above, will be indicated.

2. The extension of the rotation group

The discussion of angular momentum in quantum mechanics, usually, begins with the commutator relations for the generators of the rotation group, the components of the angular momentum operator,

$$[J_p, J_q] = i\epsilon_{pqr}, J_r \tag{11}$$

and the definition of the basis states $|j, m\rangle$ which are the simultaneous eigenstates of J^2 and J_3 belonging to the eigenvalues $j(j+1)$ and m respectively. Of special interest are the operators $J_\pm = J_x \pm iJ_y$ of which the lowering operator J_- is such that

$$J_-|j, m\rangle = [(j+m)(j-m+1)]^{1/2}|j, m-1\rangle. \tag{12}$$

It might appear that the role of J_- is analogous to that of the annihilation operator a for the oscillator. However, there is an important difference that for a given j the value of m lies in the restricted range $-j \leqslant m \leqslant +j$. This of course reflects the fact that the rotation group is compact. Thus it is not possible to find states which are eigenstates of J_- (in analogy with equation (6)). In order to build states with analogous properties, it will be necessary to introduce a group, containing the rotation group as a subgroup, which has generators which can also change the value of j. To achieve this, following Schwinger (1965), we introduce boson operators a_r $(r = +, -)$ such that

$$[a_r, a_s] = 0 = [a_r^\dagger, a_s^\dagger] \qquad \text{and} \qquad [a_r, a_s^\dagger] = \delta_{rs}. \tag{13}$$

The bilinear forms

$$J_+ = a_+^\dagger a_-, \qquad J_- = a_-^\dagger a_+ \qquad \text{and} \qquad J_3 = \tfrac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \tag{14}$$

satisfy, by virtue of the commutation relations (13), the Lie algebra of the rotation group. We may go on to consider other bilinear forms,

$$K_+ = a_+^\dagger a_-^\dagger, \qquad K_- = a_+ a_- \qquad \text{and} \qquad K_3 = \tfrac{1}{2}(a_+^\dagger a_+ + a_-^\dagger a_- + 1), \tag{15}$$

which can be seen to satisfy the commutation relations

$$[K_3, K_\pm] = \pm K_\pm \qquad \text{and} \qquad [K_+, K_-] = -2K_3. \tag{16}$$

Recognizing that the angular momentum states may be built out of the 'vacuum' by the operation of these boson creation operators, namely,

$$|j, m\rangle = \frac{(a_+^\dagger)^{j+m}(a_-^\dagger)^{j-m}}{[(j+m)!(j-m)!]^{1/2}}|0\rangle, \tag{17}$$

it may be seen that (in a manner analogous to equation (12))

$$K_-|j, m\rangle = [(j-m)(j+m)]^{1/2}|j-1, m\rangle. \tag{18}$$

In a similar manner one may introduce sets of operators: $I_+ = a_+^\dagger a_+^\dagger$, $I_- = a_+ a_+$ and $I_3 = 2a_+^\dagger a_+$; $L_+ = a_-^\dagger a_-^\dagger$, $L_- = a_- a_-$ and $L_3 = 2a_-^\dagger a_-$ which satisfy commutation relations, *mutatis mutandis*, analogous to equation (16). It may be observed that out of J_3, K_3, I_3 and L_3 only two are linearly independent, and thus out of the generators $\boldsymbol{J}$, $\boldsymbol{K}$, $\boldsymbol{I}$ and $\boldsymbol{L}$ there are ten independent generators. The action of these generators on the angular momentum states is depicted in figure 1. Thus, for instance,

$$I_-|j, m\rangle = [(j+m)(j+m-1)]^{1/2}|j-1, m-1\rangle. \tag{19}$$

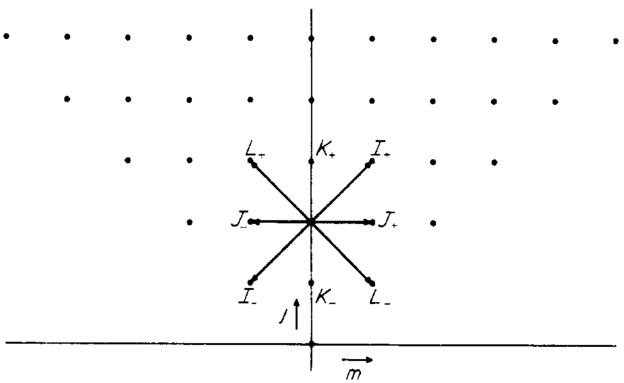

Figure 1. Action of the generators on the angular momentum states

3. Construction of angular momentum coherent states

Observing that the lowering operators I_- and K_- commute with each other and seizing upon the property of Glauber coherent states being eigenstates of the annihilation operator (see equation (6)), we introduce angular momentum coherent states as simultaneous eigenstates of I_- and K_-, namely,

$$I_-|\beta, \gamma\rangle = \beta|\beta, \gamma\rangle, \tag{20a}$$

$$K_-|\beta, \gamma\rangle = \gamma|\beta, \gamma\rangle, \tag{20b}$$

where β and γ are complex numbers. Writing

$$|\beta, \gamma\rangle = \sum_{j=0}^{\infty} \sum_{m=-j}^{+j} c_{jm}(\beta, \gamma)|j, m\rangle, \tag{21}$$

imposing conditions (20) and using equations (18) and (19), we obtain recurrence relations for the coefficients c_{jm}, which yield finally

$$|\beta, \gamma\rangle = \frac{1}{\cosh^{1/2}\xi} \sum_{j,m} \frac{\beta^m \gamma^{j-m}}{[(j+m)!(j-m)!]^{1/2}}|j, m\rangle, \tag{22}$$

where $\xi = (|\beta|^2 + |\gamma|^2)/|\beta|$, and the factor outside the summation sign is the normalization constant.

It is instructive to generate these coherent states in a somewhat different manner. In analogy to construction (7) consider states (Radcliffe 1971)

$$|\alpha; j\rangle = N\,\mathrm{e}^{\alpha J_-}|j, m = -j\rangle, \; |j, m = j\rangle \tag{23}$$

where N is a normalization constant. Such coherent states have been considered by a number of authors (Haken 1970, Haake 1973, Arecchi *et al* 1972). These so called Bloch or atomic coherent states, however, pertain to a given value of j in contradistinction to the states considered here. Next, superpose states $|\alpha; j\rangle$,

$$|\alpha; \beta\rangle = \sum_{j=0}^{\infty} c_j|\alpha; j\rangle, \tag{24}$$

such that the resultant state is an eigenstate of I_- belonging to the eigenvalue β, to obtain

$$|\alpha; \beta\rangle = \frac{1}{\cosh^{1/2}\xi} \sum_{j=0}^{\infty} \sum_{m=-j}^{+j} \frac{\alpha^{j-m}\beta^j}{[(j+m)!(j-m)!]^{1/2}}|j, m\rangle. \tag{25}$$

It is readily seen that the state defined by equation (25) is identical to the state defined by equation (22) provided we make the identification $\gamma = \alpha\beta$.

4. Properties of the angular momentum coherent states

It is readily verified that the states $|\alpha; \beta\rangle$ defined by equation (25), form a complete set of states in the sense that

$$\iint \frac{\mathrm{d}^2\alpha}{\pi} \frac{\mathrm{d}^2\beta}{\pi} \mathrm{e}^{-\xi}|\alpha; \beta\rangle\langle\alpha; \beta| = \sum_{jm} |j, m\rangle\langle j, m| = \mathbb{1}, \tag{26}$$

where the integrals cover the entire complex α and β planes.

In order to discuss the physical significance of the parameters α and β, it is appropriate to calculate the expectation values of various physical quantities in these states. Thus, for instance, we have

$$\langle\alpha, \beta|J_x|\alpha, \beta\rangle = \hbar|\beta| \operatorname{Re}\alpha \tanh \xi, \tag{27a}$$

$$\langle\alpha, \beta|J_y|\alpha, \beta\rangle = \hbar|\beta| \operatorname{Im}\alpha \tanh \xi, \tag{27b}$$

$$\langle\alpha, \beta|J_z|\alpha, \beta\rangle = \tfrac{1}{2}\xi\hbar \tanh \xi(1 - |\alpha|^2)/(1 + |\alpha|^2), \tag{27c}$$

$$\langle\alpha, \beta|J^2|\alpha, \beta\rangle = \tfrac{1}{4}\hbar^2\xi^2 + \tfrac{3}{4}\hbar^2\xi \tanh \xi. \tag{27d}$$

Introducing the parametrization

$$\alpha = e^{i\phi}\tan(\theta/2), \tag{28a}$$

$$\beta = e^{i\psi}|\beta|, \tag{28b}$$

where θ, ϕ and ψ are real parameters and passing to the classical limit

$$\hbar \to 0, \xi \to \infty: \qquad \tfrac{1}{2}\hbar\xi \to \mathscr{J}, \tag{29}$$

it is readily verified that,

$$\langle J_x\rangle \to \mathscr{J}\sin\theta\cos\phi, \tag{30a}$$

$$\langle J_y\rangle \to \mathscr{J}\sin\theta\sin\phi, \tag{30b}$$

$$\langle J_z\rangle \to \mathscr{J}\cos\theta, \tag{30c}$$

$$\langle J^2\rangle \to \mathscr{J}^2. \tag{30d}$$

Thus the physical significance of the parameters θ and ϕ resides in the co-latitude and longitude of the expectation value of the angular momentum $\boldsymbol{J}$ while the magnitude of β is related to the classical limit of the length of its expectation value. It remains to elucidate the significance of the parameter ψ. For this purpose consider the classical limits of the expectation value of a unit vector $\hat{\boldsymbol{r}}$ in this state,

$$\langle\alpha,\beta|\hat{r}_x|\alpha,\beta\rangle \to -[\cos\phi\cos\theta\cos(\psi+\phi)+\sin\phi\sin(\psi+\phi)], \tag{31a}$$

$$\langle\alpha,\beta|\hat{r}_y|\alpha,\beta\rangle \to -[\sin\phi\cos\theta\cos(\psi+\phi)-\cos\phi\sin(\psi+\phi)], \tag{31b}$$

$$\langle\alpha,\beta|\hat{r}_z|\alpha,\beta\rangle \to \sin\theta\cos(\psi+\phi). \tag{31c}$$

Thus the angle ψ is related to the nodal angle between the space-fixed and body-fixed (with the third axis along $\mathscr{J}$) coordinate systems or equivalently the azimuth of $\hat{\boldsymbol{r}}$ in the invariant plane (the plane perpendicular to $\mathscr{J}$). To sharpen the significance of these parameters further it is useful to assign the angular momentum to some physical system and consider the time dependence of various expectation values. Thus, assigning to the angular momentum state a particle of magnetic moment $g\boldsymbol{J}\hbar$ and placing the system in an external magnetic field $\boldsymbol{B}$ (say in the z direction) then each component state $|j,m\rangle$ develops with time according to the factor $\exp(-igmBt)$ which would change the coherent state with time in the manner

$$\alpha \to \alpha\, e^{-igBt}, \tag{32a}$$

$$\beta \to \beta\, e^{+igBt}, \tag{32b}$$

which is precisely what one would expect, namely, the azimuth angle ϕ for $\mathscr{J}$ changes with time with the Larmor angular frequency gB. Again it is illuminating to associate the angular momentum with that of a symmetric rigid rotator in which case the energy of the state $|j,m\rangle$ is given by

$$E_j = \frac{\hbar^2}{2\mathscr{I}}j(j+1), \tag{33}$$

where $\mathscr{I}$ is the moment of inertia parameter. In this case, in the classical limit (equation (29)) we have

$$\frac{d}{dt}\langle\hat{r}\rangle_{cl} = \boldsymbol{\omega}\times\langle\hat{r}\rangle_{cl}, \tag{34}$$

where $\boldsymbol{\omega} = \mathscr{J}/\mathscr{I}$. Thus it is clear from our discussion that the state we have constructed is 'classical' in the sense that the Glauber state was (equation (10)).

Another important question is the uncertainty relations in the states. From the commutation relations for angular momentum components we may deduce the uncertainty relation

$$\Delta J_x \, \Delta J_y \geqslant \tfrac{1}{2}\hbar|\langle J_z \rangle|, \tag{35}$$

and similarly

$$\Delta K_1 \, \Delta K_2 \geqslant \tfrac{1}{2}\hbar|\langle K_3 \rangle|, \tag{36}$$

with analogous inequalities for the components of $\boldsymbol{I}$ and $\boldsymbol{L}$. Taking expectation values in the coherent state (equation (22)) it is easily verified that

$$\langle K_1 \rangle = \mathrm{Re}\langle K_+ \rangle = \hbar \, \mathrm{Re}\, \gamma^* = \tfrac{1}{2}\xi\hbar \sin\theta \cos(\phi+\psi), \tag{37a}$$

$$\langle K_2 \rangle = -\tfrac{1}{2}\xi\hbar \sin\theta \sin(\phi+\psi), \tag{37b}$$

$$\langle K_1^2 \rangle = \tfrac{1}{4}\hbar^2 + \tfrac{1}{4}\xi\hbar^2 \tanh\xi + \tfrac{1}{4}\xi^2\hbar^2 \sin^2\theta \cos^2(\phi+\psi), \tag{37c}$$

$$\langle K_2^2 \rangle = \tfrac{1}{4}\hbar^2 + \tfrac{1}{4}\xi\hbar^2 \tanh\xi + \tfrac{1}{4}\xi^2\hbar^2 \sin^2\theta \sin^2(\phi+\psi), \tag{37d}$$

where the first two terms in the last two equations express the quantum correlations present. Thus it is clear that

$$\Delta K_1 \, \Delta K_2 = \tfrac{1}{4}\hbar^2 + \tfrac{1}{4}\xi\hbar^2 \tanh\xi. \tag{38}$$

Comparing with the result

$$\langle K_3 \rangle = \tfrac{1}{2}\hbar + \tfrac{1}{2}\xi\hbar \tanh\xi,$$

we find that in the coherent state being considered

$$\Delta K_1 \, \Delta K_2 = \tfrac{1}{2}\hbar|\langle K_3 \rangle|. \tag{39}$$

A similar result may be obtained analogously for I_1, I_2 and I_3. Thus the coherent state is a state of minimum uncertainty. However, the uncertainty relation for the components of $\mathscr{J}$ is not one of minimum uncertainty and this is not at all surprising since the magnetic projections for a given j run over a finite set and the inequality remains.

5. Conclusion

Angular momentum states analogous to the coherent states of the harmonic oscillator have been constructed, their properties have been studied. It has been shown that these states describe well the classical limits and correspond in a sense to minimum uncertainty wave packets. The usefulness of these states would perhaps lie in the consideration of statistical mechanics of spin systems, in the discussion of collective modes involving spins, and in the discussion of quantum correlation effects in near-classical systems.

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References

Arecchi F T, Courtens E, Gilmore A and Thomas H 1972 *Phys. Rev.* A 6 2211
Glauber R J 1963 *Phys. Rev* **131** 2766
Haake F 1973 *Springer Tracts*, vol 66 (Berlin: Springer)
Haken H 1970 *Encyclopedia of Physics*, vol 25/2C (Berlin: Springer)
Radcliffe J M 1971 *J. Phys.* A: *Gen. Phys.* **4** 313–23
Schrödinger E 1926 *Naturwiss.* **14** 664
Schwinger J 1965 *Quantum Theory of Angular Momentum* eds L C Biedenharn and H Van Dam (New York: Academic Press) pp 229–79